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Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Tuesday 25 June 2019

Morning (Time: 1 hour 30 minutes) | Paper Reference **9FM0/4A**

Further Mathematics
Advanced
Paper 4A: Further Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B)
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information
- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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P 6 1 1 8 3 A 0 1 2 8
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Answer ALL questions. Write your answers in the spaces provided.

1. A complex number $z = x + iy$ is represented by the point P in an Argand diagram.

Given that

$$|z - 3| = 4|z + 1|$$

(a) show that the locus of P has equation

$$15x^2 + 15y^2 + 38x + 7 = 0$$

(2)

(b) Hence find the maximum value of $|z|$

(3)

a. USE ALGEBRAIC APPROACH OF EVALUATING LOCI:

$|z-3| = 4|z+1|$

since $z = x+iy$, sub in

$|x+iy-3| = 4|x+iy+1|$

separate into real + imaginary parts

$|(x-3)+iy| = 4|(x+1)+iy|$

$\sqrt{(x-3)^2+(y)^2} = 4\sqrt{(x+1)^2+(y)^2}$

square both sides to get rid of sqrt $\sqrt{\ }$

$\left(\sqrt{(x-3)^2+(y)^2}\right)^2 = (4)^2\left(\sqrt{(x+1)^2+(y)^2}\right)^2$

$(x-3)^2+(y)^2 = 16[(x+1)^2+(y)^2]$

$x^2-6x+9+y^2 = 16[x^2+2x+1+y^2]$

$x^2-6x+9+y^2 = 16x^2+32x+16+16y^2$

$15x^2+15y^2+38x+7=0$ //

b. carrying on from part (a) ...

$15x^2+38x+15y^2+7=0$ ÷15

$x^2+\frac{38}{15}x+y^2+\frac{7}{15}=0$

$\left(x+\frac{19}{15}\right)^2-\frac{361}{225}+y^2+\frac{7}{15}=0$

rearrange to get circle eqⁿ

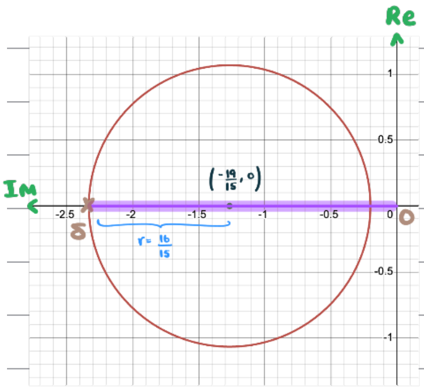
$\left(x+\frac{19}{15}\right)^2+y^2=\frac{256}{225}$

centre : $\left(-\frac{19}{15}, 0\right)$

radius : $\sqrt{\frac{256}{225}} = \frac{16}{15}$

Draw locus of P on argand diagram and look for maximum distance from O origin to a point on the circle. - We will call this furthest point S.

Question 1 continued



→ line represents $|z|$ max.

To find do centre − radius

$-\frac{19}{15} - \frac{16}{15} = -\frac{35}{15} = -\frac{7}{3}$

since $|z|_{max}$ is a length, take modulus to make +v.e.

$\left|-\frac{7}{3}\right| = \frac{7}{3}$ (length > 0)

$|z|_{max} = \frac{7}{3}$ //

(Total for Question 1 is 5 marks)

Turn over

2. The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

(a) Show that 2 is a repeated eigenvalue of **A** and find the other eigenvalue. **(5)**

(b) Hence find three non-parallel eigenvectors of **A**. **(4)**

(c) Find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. **(2)**

a. Characteristic eqn: $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}$$

$\det(A - \lambda I) = 0$

$$\det\left[\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\right] = 0$$

$$(6-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & -1 \\ 2 & 3-\lambda \end{vmatrix} + (2)\begin{vmatrix} -2 & 3-\lambda \\ 2 & -1 \end{vmatrix} = 0$$

$$(6-\lambda)\left[(3-\lambda)(3-\lambda) - (-1)(-1)\right] + (2)\left[(-2)(3-\lambda) - (2)(-1)\right] + (2)\left[(-2)(-1) - (2)(3-\lambda)\right] = 0$$

$$(6-\lambda)\left[\lambda^2 - 3\lambda - 3\lambda + 9 - 1\right] + (2)\left[-6 + 2\lambda + 2\right] + (2)\left[2 - 6 + 2\lambda\right] = 0$$

$$(6-\lambda)\left[\lambda^2 - 6\lambda + 8\right] + (2)\left[2\lambda - 4\right] + (2)\left[2\lambda - 4\right] = 0$$

$$6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

DO NOT WRITE IN THIS AREA

Question 2 continued

use factor theorem to find factor of cubic:

$f(2) = (2)^3 - 12(2)^2 + 36(2) - 32 = 0$

$\therefore (\lambda - 2)$ is a factor

$$
\begin{array}{r}
\lambda^2 - 10\lambda + 16 \\
\lambda - 2 \enclose{longdiv}{\lambda^3 - 12\lambda^2 + 36\lambda - 32} \\
- \quad \lambda^3 - 2\lambda^2 \\
-10\lambda^2 + 36\lambda \\
- \quad -10\lambda^2 + 20\lambda \\
16\lambda - 32
\end{array}
$$

$(\lambda-2)(\lambda^2 - 10\lambda + 16) = 0$

$(\lambda-2)(\lambda-8)(\lambda-2) = 0$

$(\lambda-2)^2(\lambda-8) = 0$

$\therefore$ 2 is a repeated eigenvalue & 8 is the other eigenvalue

b. $A\mathbf{x} = \lambda\mathbf{x}$

For eigenvalue 2:

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$6x - 2y + 2z = 2x$

$-2x + 3y - z = 2y$

$2x - y + 3z = 2z$

$4x - 2y + 2z = 0$

$-2x + y - z = 0$

$2x - y + z = 0$

notice that eq^n ① is $2 \times$ ②

notice that eq^n ② is $-1 \times$ ③

These 3 eq^{n}s are all multiples of each other so take just 1 eq^n and play around to find 2 eigenvectors which satisfy all 3 eq^{n}s.

P 6 1 1 8 3 A 0 5 2 8

Turn over

Question 2 continued

Take eq^n ② : $-2x + y - z = 0$

and let x = a no. e.g. $x = 1$ and find a value for y and z that work.

$-2x + y - z = 0$

$x = 1$

$y = 1$ So a possible eigenvector is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$z = -1$

If we now let:

$x = 0$

$y = 1$ so another possible eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$z = 1$

For eigenvalue 8:

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$6x - 2y + 2z = 8x$

$-2x + 3y - z = 8y$

$2x - y + 3z = 8z$

$-2x - 2y + 2z = 0$ ①

$-2x - 5y - z = 0$ ②

$2x - y - 5z = 0$ ③

① $\times -\frac{1}{2}$: $x + y - z = 0$

① − ②

$3y + 3z = 0$) $\div 3$

$y = -z$

rearrange ① : $x = z - y$

$x = z - (-z) = 2z$

← sub in $y = -z$

Question 2 continued

can now form a general eigenvector eqⁿ in terms of z : $\begin{pmatrix} 2z \\ -z \\ z \end{pmatrix}$ z can be any no.

e.g. when $z=1$, corresponding eigenvector : $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\boxed{3 \text{ non-parallel eigenvectors} : \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}$ //

c. $\boxed{P = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}}$ //   put all eigenvectors into a matrix, can be in any order

(Total for Question 2 is 11 marks)

P 6 1 1 8 3 A 0 7 2 8

4.1: Forming Recurrence Relations   4.3: Solving 2nd Order Recurrence Relations

3. The number of visits to a website, in any particular month, is modelled as the number of visits received in the previous month plus k times the number of visits received in the month before that, where k is a positive constant. $k \times V_n$

V_{n+1}

Given that V_n is the number of visits to the website in month n,

(a) write down a general recurrence relation for V_{n+2} in terms of V_{n+1}, V_n and k.

(1)

For a particular website you are given that

- $k = 0.24$ ①
- In month 1, there were 65 visits to the website. ②
- In month 2, there were 71 visits to the website. ③

(b) Show that

$$V_n = 50(1.2)^n - 25(-0.2)^n$$

(5)

This model predicts that the number of visits to this website will exceed one million for the first time in month N.

(c) Find the value of N.

(2)

a. $V_{n+2} = V_{n+1} + kV_n$ //

b. ① $V_{n+2} = V_{n+1} + 0.24V_n$

② $V_1 = 65$

③ $V_2 = 71$

solve recurrence relation ①

auxillary eq^n: $r^2 - r - 0.24 = 0$

$r^2 - r - 0.24 = 0$

$(r-1.2)(r+0.2) = 0$

$r = 1.2$ or $r = -0.2$

$V_n = A(1.2)^n + B(-0.2)^n$

sub in $n=1$ and $n=2$ to form 2 eq^{n}s and solve simultaneously

Question 3 continued

$V_1 = 65 = A(1.2)^1 + B(-0.2)^1$ ①

$V_2 = 71 = A(1.2)^2 + B(-0.2)^2$ ②

$1.2A - 0.2B = 65$ ①

$1.44A + 0.04B = 71$ ②

② × 5

$7.2A + 0.2B = 355$

② + ①

$8.4A = 420$

$A = 50$

sub A in to find B

$1.2(50) - 0.2B = 65$

$\frac{1.2(50) - 65}{0.2} = B = -25$

$A = 50, \quad B = -25$

$V_n = 50(1.2)^n - 25(-0.2)^n$

c. try subbing in values of n to satisfy inequality below:

$1000000 < 50(1.2)^n - 25(-0.2)^n$

when $n = 54$, $50(1.2)^{54} - 25(-0.2)^{54} = 943533 < 1000000$ ✗

$n = 55$, $50(1.2)^{55} - 25(-0.2)^{55} = 1132240 > 1000000$ ✓

$\therefore N = 55$

(Total for Question 3 is 8 marks)

4. (i) Use Fermat's Little Theorem to find the least positive residue of 6^{542} modulo 13
(5)

(ii) Seven students, Alan, Brenda, Charles, Devindra, Enid, Felix and Graham, are attending a concert and will sit in a particular row of 7 seats. Find the number of ways they can be seated if

(a) there are no restrictions where they sit in the row,
(1)

(b) Alan, Enid, Felix and Graham sit together,
(2)

(c) Brenda sits at one end of the row and Graham sits at the other end of the row,
(2)

(d) Charles and Devindra do not sit together.
(2)

i. Fermat's Little Theorem: $a^{13-1} = a^{12} \equiv 1 \pmod{13}$
is a prime no.

$\therefore 6^{12} \equiv 1 \pmod{13}$

$6^{542} = (6^{12})^{45} \times 6^2$
$(6^{12})^{45} \times 6^2 \pmod{13}$
$= (1)^{45} \times 6^2 \pmod{13}$
$= 36 \pmod{13}$
$= 10 \pmod{13}$

$10 \pmod{13}$

iia. no restrictions – 7 choices for 1st seat, 6 choices for 2nd seat, 5 choices for 3rd seat and ...
$\therefore 7!$
5040

b. Treat Alan, Enid, Felix, Graham as a 'single block'
This can be arranged in any order so $4!$
This 'single block', along with 3 other students can be in any order
$4!$

$4! \times 4!$
576

Question 4 continued

c. B ☐ ☐ ☐ ☐ ☐ G

OR

G ☐ ☐ ☐ ☐ ☐ B

To sort these 5, 5!

Brenda can sit on left and Graham on right or vice versa - 2 ways of doing this

$5! \times 2$

240

d. Total no. of arrangements without any restrictions: 5040 from part (a)

No. of arrangements where Charles and Devinara sit together:

'Treat Charles & Devinara as a 'single block'

$2 \times 6! = 1440$

within block 2 ways C & D can switch places (x 2)

if C & D counted as 1 block, group now has 6 people - can be arranged in 6! different ways (6!)

$5040 - 1440$

$= 3660$

3660

(Total for Question 4 is 12 marks)

P 6 1 1 8 3 A 0 1 1 2 8

5.

$$I_n = \int \operatorname{cosec}^n x \, dx \qquad n \in \mathbb{Z}$$

(a) Prove that, for $n \geqslant 2$

$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}$$

(4)

(b) Hence show that

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^6 x \, dx = \frac{56}{135}\sqrt{3}$$

(4)

a. $I_n = \int \operatorname{cosec}^n x \, dx$ split into $\operatorname{cosec}^2 x$ since we

$I_n = \int \underbrace{\operatorname{cosec}^2 x}_{v'} \underbrace{\operatorname{cosec}^{n-2} x}_{u} \, dx$ know how to integrate this from formulae booklet

$u = \operatorname{cosec}^{n-2} x$ $\qquad v' = \operatorname{cosec}^2 x$

$u' = (n-2)\operatorname{cosec}^{n-3} x \, (-\cot x \operatorname{cosec} x)$ $\qquad v = -\cot x$

Integration by parts: $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ Given in formulae booklet

$I_n = [-\cot x \operatorname{cosec}^{n-2} x] - \int (n-2)\operatorname{cosec}^{n-3} x \operatorname{cosec} x \cot^2 x \, dx$

factor out $(n-2)$ as it is a constant

$I_n = -\cot \operatorname{cosec}^{n-2} x - (n-2) \int \cot^2 x \operatorname{cosec}^{n-2} x \, dx$ use identity $1 + \cot^2 x = \operatorname{cosec}^2 x$ to replace $\cot^2 x$

$I_n = -\cot \operatorname{cosec}^{n-2} x - (n-2) \int (\operatorname{cosec}^2 x - 1) \operatorname{cosec}^{n-2} x \, dx$ expand out brackets

$I_n = -\cot \operatorname{cosec}^{n-2} x - (n-2) \int \operatorname{cosec}^n x - \operatorname{cosec}^{n-2} x \, dx$ split this integral into 2

$I_n = -\cot \operatorname{cosec}^{n-2} x - (n-2) \int \operatorname{cosec}^n x \, dx + (n-2) \int \operatorname{cosec}^{n-2} x \, dx$

$I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$ solve for I_n

$I_n (1 + (n-2)) = (n-2) I_{n-2} - \cot x \operatorname{cosec}^{n-2} x$

$I_n (n-1) = (n-2) I_{n-2} - \cot x \operatorname{cosec}^{n-2} x$ $\div (n-1)$ on both sides

$I_n = \frac{(n-2) I_{n-2} - \cot x \operatorname{cosec}^{n-2} x}{n-1}$

$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}$ //

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Question 5 continued

b. Q asking to find I_6 with: lower limit $= \frac{\pi}{3}$, upper limit $= \frac{\pi}{2}$

$$I_6 = \frac{6-2}{6-1} I_{6-2} - \frac{\operatorname{cosec}^{6-2} x \cot x}{6-1} = \frac{4}{5} I_4 - \left[\frac{\operatorname{cosec}^4 x \cot x}{5}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$I_4 = \frac{4-2}{4-1} I_{4-2} - \frac{\operatorname{cosec}^{4-2} x \cot x}{4-1} = \frac{2}{3} I_2 - \left[\frac{\operatorname{cosec}^2 x \cot x}{3}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

write eqⁿs in terms of I_6, I_4, I_2 then find these individually and sub back in

Work out I_2 directly:

$$I_2 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^2 x \, dx = \left[-\cot x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \left[\left\{-\cot\left(\tfrac{\pi}{2}\right)\right\} - \left\{-\cot\left(\tfrac{\pi}{3}\right)\right\}\right]$$

$$= \left[(0) - \left(-\frac{\sqrt{3}}{3}\right)\right] = \frac{\sqrt{3}}{3}$$

$$I_2 = \frac{\sqrt{3}}{3}$$

$$I_4 = \frac{2}{3}\left(\frac{\sqrt{3}}{3}\right) - \left[\left(\frac{\operatorname{cosec}^2\left(\frac{\pi}{2}\right)\cot\left(\frac{\pi}{2}\right)}{3}\right) - \left(\frac{\operatorname{cosec}^2\left(\frac{\pi}{3}\right)\cot\left(\frac{\pi}{3}\right)}{3}\right)\right]$$

$$= \frac{2\sqrt{3}}{9} - \left(-\frac{4\sqrt{3}}{27}\right)$$

$$= \frac{10\sqrt{3}}{27}$$

$$I_6 = \frac{4}{5}\left(\frac{10\sqrt{3}}{27}\right) - \left[\left(\frac{\operatorname{cosec}^4\left(\frac{\pi}{2}\right)\cot\left(\frac{\pi}{2}\right)}{5}\right) - \left(\frac{\operatorname{cosec}^4\left(\frac{\pi}{3}\right)\cot\left(\frac{\pi}{3}\right)}{5}\right)\right]$$

$$= \frac{8\sqrt{3}}{27} - \left(-\frac{16\sqrt{3}}{135}\right)$$

$$= \frac{56\sqrt{3}}{135} \,//$$

$$\frac{56\sqrt{3}}{135}$$

2.1: The Axioms For A Group | 2.2: Cayley Tables & Finite Groups | 2.3: Order & Subgroups | 2.4: Isomorphism

6. (i) A binary operation $*$ is defined on positive real numbers by

$$a * b = a + b + ab$$

Prove that the operation $*$ is associative.

(4)

(ii) The set $G = \{1, 2, 3, 4, 5, 6\}$ forms a group under the operation of multiplication modulo 7

$(a \times b) \bmod 7$

(a) Show that G is cyclic.

(2)

The set $H = \{1, 5, 7, 11, 13, 17\}$ forms a group under the operation of multiplication modulo 18

(b) List all the subgroups of H.

(3)

(c) Describe an isomorphism between G and H.

(3)

i. To prove associativity: $a * (b * c) = (a * b) * c$

$(a * b) * c$
$= (a + b + ab) * c$
$= a + b + ab + c + (a + b + ab)(c)$
$= a + b + ab + c + ac + bc + abc$

$a * (b * c)$
$= a * (b + c + bc)$
$= a + b + c + bc + (b + c + bc)(a)$
$= a + b + c + bc + ab + ac + abc$

LHS = RHS $\therefore$ $*$ is associative

iia.

G:

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Question 6 continued

$2^1 = 2$	$3^1 = 3$	$4^1 = 4$	$5^1 = 5$	$6^1 = 6$
$2^2 = 4$	$3^2 = 2$	$4^2 = 2$	$5^2 = 4$	$6^2 = 1$
$2^3 = 1$	$3^3 = 6$	$4^3 = 1$	$5^3 = 6$	$6^3 = 6$
$2^4 = 2$	$3^4 = 4$	$4^4 = 4$	$5^4 = 2$	$6^4 = 1$
$2^5 = 4$	$3^5 = 5$	$4^5 = 2$	$5^5 = 3$	$6^5 = 6$
$2^6 = 1$	$3^6 = 1$	$4^6 = 1$	$5^6 = 1$	$6^6 = 1$
2 generates $\{2,4,1\}$ so order 2	3 generates entire group	4 generates $\{4,2,1\}$ so order 3	5 generates entire group	6 generates $\{6,1\}$ so order 2

3/5 have order 6 so generates the group so G is cyclic //

b. The group H has 6 elements so $|H| = 6$

By Lagrange's theorem, order of any subgroup must divide order of group H.

So possible subgroups are: 1, 2, 3, 6

Subgroup order 1: $\{1\}$ ← pick $\{1\}$ since this is identity element

Subgroup order 6: $\{1, 5, 7, 11, 13, 17\}$

H

$\times_{18}$	1	5	7	11	13	17
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

← will use to find subgroups of order 2/3.

$5^1 = 5$	$7^1 = 7$	$11^1 = 11$	$13^1 = 13$	$17^1 = 17$
$5^2 = 7$	$7^2 = 13$	$11^2 = 13$	$13^2 = 7$	$17^2 = 1$
$5^3 = 17$	$7^3 = 1$	$11^3 = 17$	$13^3 = 1$	$17^3 = 17$
$5^4 = 13$	$7^4 = 7$	$11^4 = 7$	$13^4 = 13$	$17^4 = 1$
$5^5 = 11$	$7^5 = 13$	$11^5 = 5$	$13^5 = 7$	$17^5 = 17$
$5^6 = 1$	$7^6 = 1$	$11^6 = 1$	$13^6 = 1$	$17^6 = 1$

Order 2	Order 3
17	7, 13
$\{1, 17\}$	$\{1, 7, 13\}$

Question 6 continued

subgroups: {1}, {1, 17}, {1, 7, 13}, {1, 5, 7, 11, 13, 17}

a.

	1	2	3	6
orders of G	1	6	2, 4	3, 5
orders of H	1	17	7, 13	5, 11

Based off of table above

G → H
1 → 1

G → H
6 → 17

Others must randomly plug in and see what works and what doesn't

try: = locked in

$\times_7$	G	1	2	3	4	5	6
$\times_{18}$	H	1	7	5	13	11	17

to check: $(2 \times 5) \bmod 7 = 3$
$(7 \times 11) \bmod 18 = 5$

We can see G → H so this works
3 → 5

G	1	2	3	4	5	6
H	1	7	5	13	11	17

or

G	1	2	3	4	5	6
H	1	13	11	7	5	17

mark scheme accepts either answer

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7. A transformation from the z-plane to the w-plane is given by

$$w = \frac{3\text{i}z - 2}{z + \text{i}} \qquad z \neq -\text{i}$$

(a) Show that the circle C with equation $|z + \text{i}| = 1$ in the z-plane is mapped to a circle D in the w-plane, giving a Cartesian equation for D.

(4)

(b) Sketch C and D on Argand diagrams.

(2)

a. $w = \dfrac{3iz - 2}{z + i}$

$w(z+i) = 3iz - 2$

$wz + wi = 3iz - 2$

$2 + wi = 3iz - wz$

$2 + wi = z(3i - w)$

$z = \dfrac{2 + wi}{3i - w}$

rearrange for z

sub this into $|z + i| = 1$

$\left|\dfrac{2+wi}{3i-w} + i\right| = 1$

$\left|\dfrac{2 + wi + i(3i - w)}{3i - w}\right| = 1$

$\left|\dfrac{2 + \cancel{wi} + 3i^2 - \cancel{iw}}{3i - w}\right| = 1$

$\left|\dfrac{2 - 3}{3i - w}\right| = 1$

$\left|\dfrac{-1}{3i - w}\right| = 1$

split using rule : $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

$\dfrac{|-1|}{|3i - w|} = 1$

$|-1| = 1$

$\dfrac{1}{|3i - w|} = 1$

$|3i - w| = 1$

$|(-1)(-3i + w)| = 1$

factor out -1 to make w +v.e.

Question 7 continued

$|(-1)(w-3i)| = 1$ } $|ab| = |a||b|$

$|(-1)||(w-3i)| = 1$

$1|(w-3i)| = 1$

$|w-3i| = 1$

let $w = u + vi$ (remember: $z = x + yi$ and $w = u + vi$)

$|(u+vi) - 3i| = 1$

$|(u) + (v-3)i| = 1$

$\sqrt{(u)^2 + (v-3)^2} = 1$

$\left(\sqrt{(u)^2 + (v-3)^2}\right)^2 = (1)^2$ } square both sides to get rid of sqrt $\sqrt{\ }$

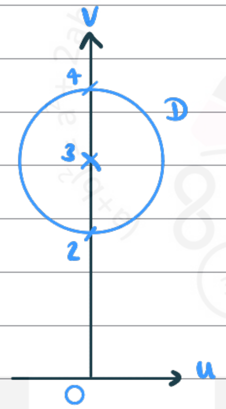
$\left(\cancel{\sqrt{\ }}(u)^2 + (v-3)^2\right)^{\cancel{2}} = (1)^2$

$u^2 + (v-3)^2 = 1$ //

b. $|w-3i| = 1$

is a circle eq^n for centre: $(0,3)$

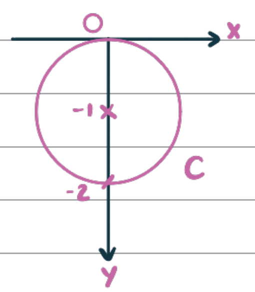
radius: 1



$|z+i| = 1$

is a circle eq^n for centre: $(0,-1)$

radius: 1



8.

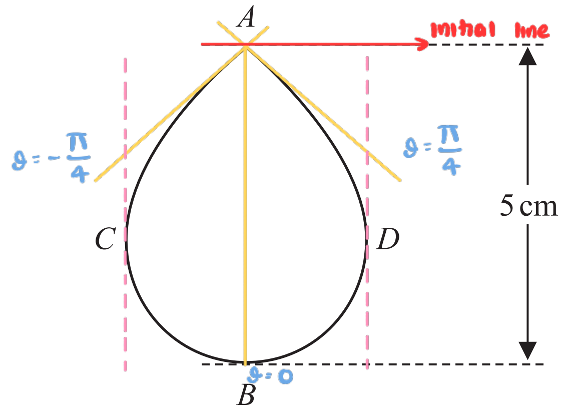


Figure 1

Figure 1 shows the vertical cross section of a child's spinning top. The point A is vertically above the point B and the height of the spinning top is 5 cm.

The line CD is perpendicular to AB such that CD is the maximum width of the spinning top.

The spinning top is modelled as the solid of revolution created when part of the curve with polar equation

$$r^2 = 25\cos 2\theta$$

is rotated through 2π radians about the initial line.

(a) Show that, according to the model, the surface area of the spinning top is

$$k\pi(2 - \sqrt{2})\,\text{cm}^2$$

where k is a constant to be determined.

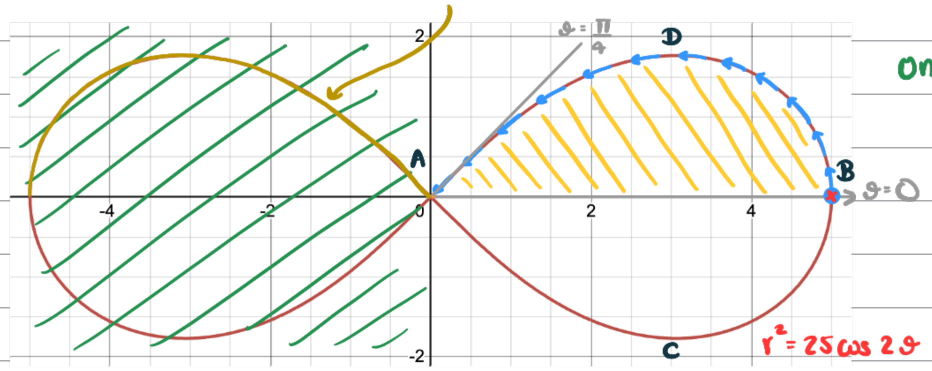
(7)

(b) Show that, according to the model, the length CD is $\dfrac{5\sqrt{2}}{2}$ cm.

(6)

a.

ϑ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	5	$\frac{5\sqrt{2}}{2}$	0	X	X	X	0	$\frac{5\sqrt{2}}{2}$	5



Only focus on 1 curve

Since surface area already considers symmetry, no doubling is necessary.

$\therefore$ limits are $\vartheta = 0$ and $\vartheta = \frac{\pi}{4}$

P 6 1 1 8 3 A 0 2 4 2 8

Question 8 continued

a. $S_x = 2\pi \int r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ GIVEN IN FORMULAE BOOKLET

→ find $\frac{dr}{d\theta}$, then find $\left(\frac{dr}{d\theta}\right)^2$:

$r^2 = 25\cos 2\theta$

$2r\left(\frac{dr}{d\theta}\right) = -50\sin 2\theta$

$\frac{dr}{d\theta} = -\frac{50\sin 2\theta}{2r} = -\frac{25}{r}\sin 2\theta$

$\left(\frac{dr}{d\theta}\right)^2 = \left(-\frac{25}{r}\sin 2\theta\right)^2 = \frac{625}{r^2}\sin^2 2\theta$

$\left(\frac{dr}{d\theta}\right)^2 = \frac{625}{r^2}\sin^2 2\theta$

sub back into formulae

$S_x = 2\pi \int_0^{\pi/4} r\sin\theta \sqrt{r^2 + \frac{625\sin^2 2\theta}{r^2}}\, d\theta$

factor out $\left(\frac{1}{r^2}\right)$

$S_x = 2\pi \int_0^{\pi/4} r\sin\theta \sqrt{\frac{1}{r^2}\left(r^4 + 625\sin^2 2\theta\right)}\, d\theta$

split surd into 2 and simplify $\sqrt{\frac{1}{r^2}}$

$S_x = 2\pi \int_0^{\pi/4} r\sin\theta \sqrt{\frac{1}{r^2}} \sqrt{\left(r^4 + 625\sin^2 2\theta\right)}\, d\theta$

$S_x = 2\pi \int_0^{\pi/4} \cancel{r}\sin\theta \left(\frac{1}{\cancel{r}}\right) \sqrt{\left(r^4 + 625\sin^2 2\theta\right)}\, d\theta$

$S_x = 2\pi \int_0^{\pi/4} \sin\theta \sqrt{\left(r^4 + 625\sin^2 2\theta\right)}\, d\theta$

Must convert r^4 to θ in order to integrate - use original polar eq^n

$r^2 = 25\cos 2\theta$

$r^4 = (r^2)^2 = (25\cos 2\theta)^2 = 625\cos^2 2\theta$

$r^4 = 625\cos^2 2\theta$

$S_x = 2\pi \int_0^{\pi/4} \sin\theta \sqrt{\left(625\cos^2 2\theta + 625\sin^2 2\theta\right)}\, d\theta$

$S_x = 2\pi \int_0^{\pi/4} \sin\theta \sqrt{625\left(\cos^2 2\theta + \sin^2 2\theta\right)}\, d\theta$ trig identity: $\sin^2(\theta) + \cos^2(\theta) = 1$

$S_x = 2\pi \int_0^{\pi/4} \sin\theta \sqrt{625\,(1)}\, d\theta$

Question 8 continued

$$Sx = 2\pi \int_0^{\pi/4} \sin\theta\,(25)\,d\theta$$

take 25 out integral

$$Sx = 50\pi \int_0^{\pi/4} \sin\theta\,d\theta$$

$$Sx = 50\pi \left[-\cos\theta \right]_0^{\frac{\pi}{4}}$$

$$Sx = 50\pi \left[\left(-\cos\frac{\pi}{4}\right) - \left(-\cos 0\right) \right]$$

$$Sx = 50\pi \left(\frac{2-\sqrt{2}}{2} \right) = 25\pi(2-\sqrt{2})\ \text{cm}^2$$

$K = 25$

b. since C and D a perpendicular, find $\frac{dy}{d\theta} = 0$.

$$y = r\sin\theta$$
$$y^2 = r^2\sin^2\theta$$
replace $r^2 = 25\cos 2\theta$
$$y^2 = 25\cos 2\theta \sin^2\theta$$

$$2y\left(\frac{dy}{d\theta}\right) = -50\sin 2\theta \sin^2\theta + 50\cos 2\theta \sin\theta\cos\theta$$

differentiate both sides w.r.t. θ

$y = r\sin\theta$ (convert into eq^n of just θ)

$$2r\sin\theta\left(\frac{dy}{d\theta}\right) = -50\sin 2\theta\sin^2\theta + 50\cos 2\theta\sin\theta\cos\theta$$

divide both sides by $2r\sin\theta$

$$\frac{dy}{d\theta} = \frac{-50\sin 2\theta\sin^2\theta + 50\cos 2\theta\sin\theta\cos\theta}{2r\sin\theta}$$

$$\frac{dy}{d\theta} = \frac{-\overset{25}{\cancel{50}}\sin 2\theta\sin^{\cancel{2}}\theta}{\cancel{2}r\cancel{\sin\theta}} + \frac{\overset{25}{\cancel{50}}\cos 2\theta\cancel{\sin\theta}\cos\theta}{\cancel{2r\sin\theta}}$$

$$\frac{dy}{d\theta} = \frac{-25\sin 2\theta\sin\theta}{r} + \frac{25\cos 2\theta\cos\theta}{r}$$

DO NOT WRITE IN THIS AREA

Question 8 continued

$$\frac{dy}{d\theta} = \frac{-25\sin 2\theta \sin\theta + 25\cos 2\theta \cos\theta}{r}$$

replace $\frac{dy}{d\theta} = 0$

$$0 = \frac{-25\sin 2\theta \sin\theta + 25\cos 2\theta \cos\theta}{r}$$

multiply both sides by r $(\times r)$

$$0 = -25\sin 2\theta \sin\theta + 25\cos 2\theta \cos\theta$$

$$0 = -25(2\sin\theta\cos\theta)\sin\theta + 25(2\cos^2\theta - 1)\cos\theta$$

$$0 = -25(2\sin^2\theta\cos\theta) + 50\cos^3\theta - 25\cos\theta$$

$$0 = -50(1-\cos^2\theta)\cos\theta + 50\cos^3\theta - 25\cos\theta$$

$$0 = -50\cos\theta + 50\cos^3\theta + 50\cos^3\theta - 25\cos\theta$$

$$0 = 100\cos^3\theta - 75\cos\theta$$

$$0 = \cos\theta(100\cos^2\theta - 75)$$

use double angle formulae + trig identities to create eq^n of only cos terms

$\cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$100\cos^2\theta - 75 = 0$

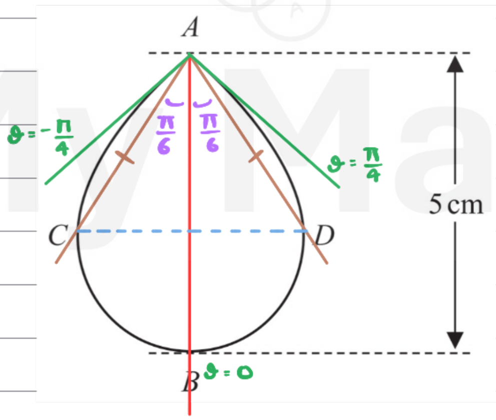
$\cos^2\theta = \frac{75}{100} = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6}, -\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{11\pi}{6}$

However, can only accept $\theta = \frac{\pi}{6}, \theta = -\frac{\pi}{6}$ as others graphically don't work

$\llcorner -\frac{\pi}{4} < \theta < \frac{\pi}{4}$



must work out $|AD|$:

$r^2 = 25\cos(2\times\frac{\pi}{6}) = \frac{25}{2}$

$r = +\sqrt{\frac{25}{2}} = \frac{5\sqrt{2}}{2}$

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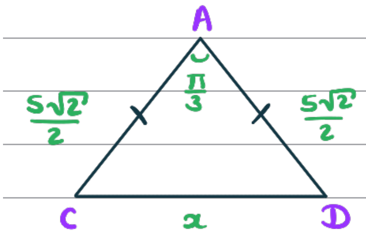
Question 8 continued



use cosine rule:

$x^2 = \left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{2}\right)^2 - (2)\left(\frac{5\sqrt{2}}{2}\right)\left(\frac{5\sqrt{2}}{2}\right)\cos\left(\frac{\pi}{3}\right)$

$x^2 = \frac{25}{2}$

$x = \frac{5\sqrt{2}}{2}$ (only take +v.e. value since length cannot be -v.e.)

$\therefore |CD| = \frac{5\sqrt{2}}{2}$ cm //

(Total for Question 8 is 13 marks)

TOTAL FOR PAPER IS 75 MARKS